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Optimal lower bounds on the hydrostatic stress amplification inside random two-phase elastic composites

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Abstract

Composites made from two linear isotropic elastic materials are subjected to a uniform hydrostatic stress. It is assumed that only the volume fraction of each elastic material is known. Lower bounds on all rth moments of the hydrostatic stress field inside each phase are obtained for $r \geqslant 2$. A lower bound on the maximum value of the hydrostatic stress field is also obtained. These bounds are given by explicit formulas depending on the volume fractions of the constituent materials and their elastic moduli. All of these bounds are shown to be the best possible as they are attained by the hydrostatic stress field inside the Hashin–Shtrikman coated sphere assemblage. The bounds provide a new opportunity for the assessment of load transfer between macroscopic and microscopic scales for statistically defined microstructures.

Keywords: Composite materials; Failure criteria; Stress concentrations

1. Introduction

Many composite structures are hierarchical in nature and are made up of substructures distributed across several length scales. Examples include fiber-reinforced

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laminates as well as naturally occurring structures like bone. From the perspective of failure initiation, it is crucial to quantify the load transfer between length scales. It is common knowledge that the load transfer can result in local stresses that are significantly greater than the applied macroscopic stress, see for example Kelly and Macmillan (1986). Quantities useful for the study of load transfer include the higher-order moments of the local stress. The higher moments are sensitive to local stress concentrations generated by the interaction between the microstructure and the macroscopic load. In this article, we find optimal lower bounds on all higher moments of the local hydrostatic stress inside random composites made from two isotropic elastic materials in prescribed proportions. These bounds provide the minimum amount of hydrostatic stress amplification that can be expected from this class of composites. Composite systems that are sensitive to local hydrostatic stress concentrations include rubber toughened plastics, see Chen and Mai (1999).

The composite is subjected to a macroscopic hydrostatic loading $\sigma_0 I$, where σ_0 is a constant and I is the identity matrix. The local stress tensor at each point $\mathbf{x}$ inside the composite is denoted by $\sigma_{ij}(\mathbf{x})$ and the hydrostatic stress $\sigma_{\mathrm{H}}(\mathbf{x})$ is given by

$$\sigma_{\mathrm{H}}(\mathbf{x}) = (\sigma_{11}(\mathbf{x}) + \sigma_{22}(\mathbf{x}) + \sigma_{33}(\mathbf{x}))/3. \tag{1.1}$$

For planar elastic problems the hydrostatic stress reduces to $\sigma_{\mathrm{H}}(\mathbf{x}) = (\sigma_{11}(\mathbf{x}) + \sigma_{22}(\mathbf{x}))/2$. The load transfer is expressed by the ratio $R_{\mathrm{H}}(\mathbf{x})$ relating the local hydrostatic stress $\sigma_{\mathrm{H}}(\mathbf{x})$ to the imposed macroscopic stress σ_0 given by

$$R_{\mathrm{H}}(\mathbf{x}) = \frac{\sigma_{\mathrm{H}}(\mathbf{x})}{\sigma_0}. \tag{1.2}$$

In this treatment, both two- and three-dimensional elastic problems are considered. For three-dimensional problems the composite is contained inside a unit cube Q. The two-dimensional elastic problem corresponds to long fiber reinforced materials with fixed cross-sectional geometry subjected to transverse loads. In this context, Q is a unit square containing a transverse slice of the fibrous composite. The volume or area average of a quantity q over Q is denoted by $\langle q \rangle$.

In this work, we consider the moments of the load transfer ratio $R_{\mathrm{H}}(\mathbf{x})$ inside each phase given by

$$\langle \chi_1 |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r} \quad \text{and} \quad \langle \chi_2 |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r} \tag{1.3}$$

for $2 \leqslant r < \infty$. Here, χ_1 is the indicator function of material one taking the value 1 inside the set occupied by material one and 0 outside. Similarly χ_2 is the indicator function of material two and $\chi_2(\mathbf{x}) = 1 - \chi_1(\mathbf{x})$. The part of Q occupied by material one is denoted by Q_1 and the part occupied by material two is denoted by Q_2. We also consider the L^∞ norms given by

$$\begin{aligned} \|R_{\mathrm{H}}\|_{L^\infty(Q_1)} &= \lim_{r\to\infty} \langle \chi_1 |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r}, \\ \|R_{\mathrm{H}}\|_{L^\infty(Q_2)} &= \lim_{r\to\infty} \langle \chi_2 |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r}, \\ \|R_{\mathrm{H}}\|_{L^\infty(Q)} &= \lim_{r\to\infty} \langle |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r}. \end{aligned} \tag{1.4}$$

The moments and L^∞ norms of R_H provide measures of the load transfer that are sensitive to the effects of local stress amplification generated by the microgeometry. Explicit optimal lower bounds on the moments (1.3) and L^∞ norms (1.4) are derived. These bounds hold for all configurations of the two elastic materials subject to prescribed constraints on the volume fractions of each of the materials. It is shown that the configurations that minimize all moments and L^∞ norms of R_H are given by the Hashin and Shtrikman (1962) coated sphere assemblages. For the two-dimensional problem the optimal configurations are given by the Hashin and Shtrikman (1962) coated cylinder assemblages. These configurations are described in detail in Section 5. The approach presented here is motivated by the observations recently used to obtain optimal lower bounds on the higher moments and the L^∞ norm of the electric field for two-phase random dielectrics, see Lipton (2004).

We conclude by noting that earlier work identifies the optimal inclusion shapes that minimize the maximum eigenvalue of the local stress for a given constant applied stress. These investigations are carried out in the context of two-phase linear elasticity. The work presented in Wheeler (1993) provides an optimal lower bound on the supremum of the maximum principal stress for a single simply connected stiff inclusion in an infinite matrix subject to a remote stress at infinity. The optimal shapes are given by ellipsoids. The work presented in Grabovsky and Kohn (1995) provides an optimal lower bound on the supremum of the maximum principal stress for two-dimensional periodic composites consisting of a single simply connected stiff inclusion in the period cell. The bound is given in terms of the area fraction of the included phase and for an explicit range of prescribed average stress the optimal inclusions are given by Vigdergauz (1994) shapes.

2. Elastic boundary value problem for composite materials

In this section, we recall the canonical boundary value problem used to describe elastic fields in composite materials, see for example Milton (2002). The elastic stress and strain fields $\sigma(\mathbf{x})$ and $\varepsilon(\mathbf{x})$ inside the composite satisfy $\varepsilon_{ij}(\mathbf{x}) = (u_{i,j}(\mathbf{x}) + u_{j,i}(\mathbf{x}))/2$ and $\sigma(\mathbf{x}) = C(\mathbf{x})\varepsilon(\mathbf{x})$. Here, $C(\mathbf{x})$ is the local elasticity tensor and $u_{i,j}$ is the derivative of the ith component of the displacement along the jth direction. The elasticity tensors of materials one and two are specified by the shear and bulk moduli μ^1, κ^1 and μ^2, κ^2 respectively. Without loss of generality it is supposed that $\mu^1 > \mu^2$. The equation of elastic equilibrium inside each phase is given by

$$\operatorname{div} \sigma = 0. \tag{2.1}$$

It is assumed that there is perfect bonding between the materials so that the displacement $\mathbf{u}$ and traction $\sigma\mathbf{n}$ are continuous across the two phase interface, i.e.,

$$\begin{aligned} \mathbf{u}_{|_1} &= \mathbf{u}_{|_2}, \\ \sigma_{|_1}\mathbf{n} &= \sigma_{|_2}\mathbf{n}. \end{aligned} \tag{2.2}$$

Here, $\mathbf{n}$ is the unit normal to the interface pointing into material two and the subscripts indicate the side of the interface that the displacement and traction fields are evaluated on. It is supposed that Q is the period cell for an infinite elastic medium and the elastic displacement $\mathbf{u}$ inside the composite is decomposed into a Q periodic part $\mathbf{u}^{\mathrm{per}}$ and a linear part $\overline{\varepsilon}_{ij}\mathbf{x}_j$ such that $\mathbf{u}_i = \mathbf{u}_i^{\mathrm{per}} + \overline{\varepsilon}_{ij}\mathbf{x}_j$. It is easily seen that $\overline{\varepsilon} = \langle \varepsilon \rangle$. The composite is subjected to an imposed average macroscopic stress $\overline{\sigma} = \langle \sigma \rangle$. The effective elastic tensor C^{e} relates the imposed macroscopic stress to the average strain and is given by

$$\overline{\sigma} = C^{\mathrm{e}}\overline{\varepsilon}. \tag{2.3}$$

In this paper, the imposed macroscopic stress is taken to be hydrostatic, i.e.,

$$\overline{\sigma} = \sigma_0 I. \tag{2.4}$$

The optimal lower bounds presented in subsequent sections will be established with the aid of the following identities relating the effective elastic properties to the first moment of the hydrostatic stress in each phase. The identities are given by

$$\mathrm{tr}\{\langle \chi_2(\mathbf{x})\sigma(\mathbf{x})\rangle\} = \frac{d\sigma_0\kappa^2(1-\kappa^1(C^{\mathrm{e}})^{-1}I:I)}{\kappa^2-\kappa^1} \tag{2.5}$$

and

$$\mathrm{tr}\{\langle \chi_1(\mathbf{x})\sigma(\mathbf{x})\rangle\} = \frac{d\sigma_0\kappa^1(1-\kappa^2(C^{\mathrm{e}})^{-1}I:I)}{\kappa^1-\kappa^2}. \tag{2.6}$$

We establish the first identity noting that the second follows from identical arguments. Expanding Eq. (2.3) one obtains

$$C^{\mathrm{e}}\overline{\varepsilon} = \langle (C^1 + \chi_2(C^2 - C^1))\varepsilon(\mathbf{x})\rangle. \tag{2.7}$$

Rearranging terms and taking the trace gives

$$\frac{\kappa^2}{\kappa^2-\kappa^1}\,\mathrm{tr}\{\overline{\eta}\} = \mathrm{tr}\{\langle \chi_2(\mathbf{x})\sigma(\mathbf{x})\rangle\}, \tag{2.8}$$

where

$$\overline{\eta} = (C^{\mathrm{e}} - C^1)\overline{\varepsilon} = (C^{\mathrm{e}} - C^1)(C^{\mathrm{e}})^{-1}\overline{\sigma}. \tag{2.9}$$

Upon setting $\overline{\sigma} = \sigma_0 I$ a straightforward calculation delivers Eq. (2.5).

3. Optimal lower bounds on the load transfer ratio

Optimal lower bounds on the moments and L^∞ norms of the load transfer ratio R_{H} are presented. The optimal bounds are given in terms of the proportions and the elastic constants of the two materials. For future reference the proportions of material one and two are specified by θ_1 and θ_2, respectively, and are given by the volume or area averages of the characteristic functions

$$\theta_1 = \langle \chi_1 \rangle, \quad \theta_2 = \langle \chi_2 \rangle, \quad \theta_1 + \theta_2 = 1. \tag{3.1}$$

In what follows, the parameter d gives the dimensionality of the elastic problem under consideration. The particular form of the lower bounds depends on whether the elastic materials are well ordered, $\kappa^1 > \kappa^2$ or non-well-ordered $\kappa^1 < \kappa^2$.

We introduce the stress amplification factors given by

$$L^1 = \frac{\kappa^1(\kappa^2 + 2\mu^2(d-1)/d)}{\kappa^1\kappa^2 + (\theta_1\kappa^1 + \theta_2\kappa^2)2\mu^2(d-1)/d}, \tag{3.2}$$

$$L^2 = \frac{\kappa^2(\kappa^1 + 2\mu^1(d-1)/d)}{\kappa^1\kappa^2 + (\theta_1\kappa^1 + \theta_2\kappa^2)2\mu^1(d-1)/d}, \tag{3.3}$$

$$M^1 = \frac{\kappa^1(\kappa^2 + 2\mu^1(d-1)/d)}{\kappa^1\kappa^2 + (\theta_1\kappa^1 + \theta_2\kappa^2)2\mu^1(d-1)/d}, \tag{3.4}$$

and

$$M^2 = \frac{\kappa^2(\kappa^1 + 2\mu^2(d-1)/d)}{\kappa^1\kappa^2 + (\theta_1\kappa^1 + \theta_2\kappa^2)2\mu^2(d-1)/d}. \tag{3.5}$$

The lower bounds and the associated optimal configurations are presented in the following two subsections.

3.1. *Optimal lower bounds for the well-ordered case $\kappa^1 > \kappa^2$*

For the well-ordered case $\kappa^1 > \kappa^2$ and $L^1 > 1 > L^2$. The optimal lower bounds on the moments of R_{H} are given by the following results.

3.1.1. *Optimal lower bounds on the moments of the load transfer ratio in material one*

For fixed values of θ_1 and θ_2, the load transfer ratio inside material one satisfies

$$\theta_1^{1/r} L^1 \leqslant \langle \chi_1 |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r} \quad \text{for } 2 \leqslant r \leqslant \infty. \tag{3.6}$$

Moreover, for $d = 2(3)$, the load transfer ratio inside material one for the coated cylinder (sphere) assemblage with core of material one and coating of material two attains the lower bound (3.6) for every r in $2 \leqslant r \leqslant \infty$.

3.1.2. *Optimal lower bounds on the moments of the load transfer ratio in material two*

For fixed values of θ_1 and θ_2, the load transfer ratio inside material two satisfies

$$\theta_2^{1/r} L^2 \leqslant \langle \chi_2 |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r} \quad \text{for } 2 \leqslant r \leqslant \infty. \tag{3.7}$$

Moreover, for $d = 2(3)$, the load transfer ratio inside material two for the coated cylinder (sphere) assemblage with core of material two and coating of material one attains the lower bound (3.7) for every r in $2 \leqslant r \leqslant \infty$.

3.1.3. *Optimal lower bound on the L^∞ norm of the load transfer ratio*

For fixed values of θ_1 and θ_2, the load transfer ratio satisfies

$$L^1 \leqslant \| |R_{\mathrm{H}}(\mathbf{x})| \|_{L^\infty(Q)}. \tag{3.8}$$

Moreover, for $d = 2(3)$, the load transfer ratio inside the coated cylinder (sphere) assemblage with core of material one and coating of material two attains the lower bound (3.8).

For $\kappa_1 > \kappa_2$ and $1 > \theta_1 > 0$ it follows from Eq. (3.8) that composites amplify the imposed macroscopic stress. Indeed, since $L^1 > 1$ for $\kappa_1 > \kappa_2$ and $1 > \theta_1 > 0$, it is clear that every composite has points inside material one where the magnitude of the local hydrostatic stress is greater than the magnitude of the applied macroscopic stress. The stress amplification given by L^1 is plotted in Fig. 1 as a function of θ_1 and κ^2/κ^1 for $\kappa^2/\mu^2 = \frac{1}{3}$ and $0.1 \leqslant \theta_1 \leqslant 1$.

3.2. Optimal lower bounds for the non-well-ordered case $\kappa^1 < \kappa^2$

For $\kappa^1 < \kappa^2$, one has that $M^2 > 1 > M^1$. The optimal lower bounds on the moments of R_H are given by the following results.

3.2.1. Optimal lower bounds on the moments of the load transfer ratio in material one

For fixed values of θ_1 and θ_2 the load transfer ratio inside material one satisfies

$$\theta_1^{1/r} M^1 \leqslant \langle \chi_1 |R_H(\mathbf{x})|^r \rangle^{1/r} \quad \text{for } 2 \leqslant r \leqslant \infty. \tag{3.9}$$

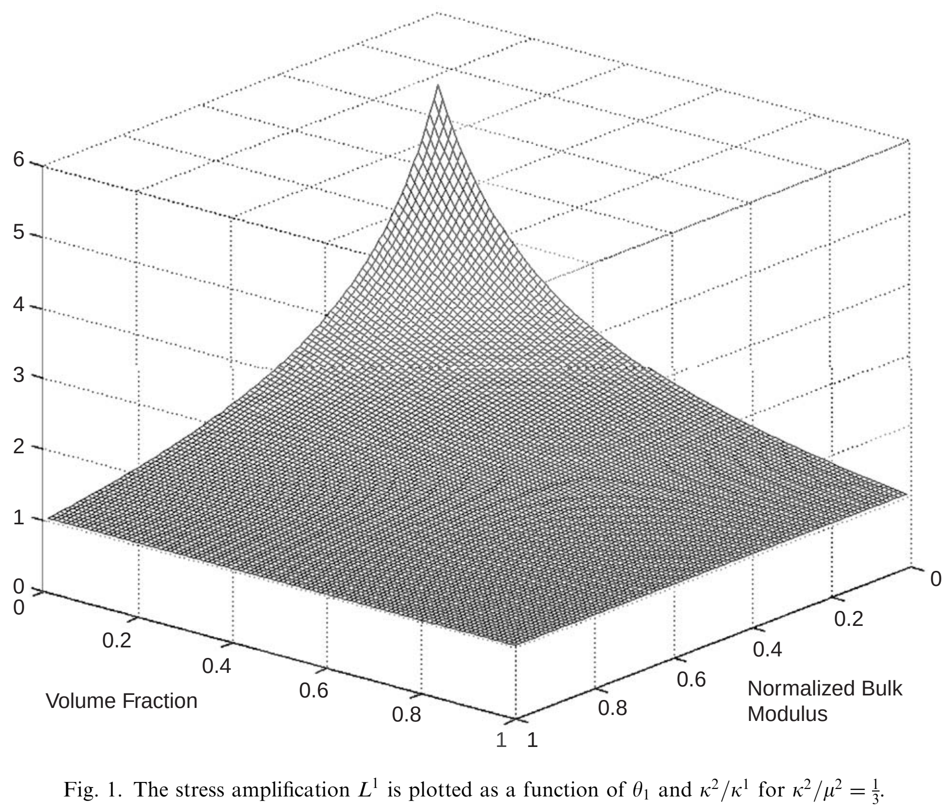

Fig. 1. The stress amplification L^1 is plotted as a function of θ_1 and κ^2/κ^1 for $\kappa^2/\mu^2 = \frac{1}{3}$.

Moreover, for $d = 2(3)$, the load transfer ratio inside material one for the coated cylinder (sphere) assemblage with core of material two and coating of material one attains the lower bound (3.9) for every r in $2 \leqslant r \leqslant \infty$.

3.2.2. Optimal lower bounds on the moments of the load transfer ratio in material two

For fixed values of θ_1 and θ_2, the load transfer ratio inside material two satisfies

$$\theta_2^{1/r} M^2 \leqslant \langle \chi_2 |R_{\mathrm{H}}(\mathbf{x})|^r \rangle^{1/r} \quad \text{for } 2 \leqslant r \leqslant \infty. \tag{3.10}$$

Moreover, for $d = 2(3)$, the load transfer ratio inside material two for the coated cylinder (sphere) assemblage with core of material one and coating of material two attains the lower bound (3.10) for every r in $2 \leqslant r \leqslant \infty$.

3.2.3. Optimal lower bound on the L^∞ norm of the load transfer ratio

For fixed values of θ_1 and θ_2, the load transfer ratio satisfies

$$M^2 \leqslant \| |R_{\mathrm{H}}(\mathbf{x})| \|_{L^\infty(Q)}. \tag{3.11}$$

Moreover, for $d = 2(3)$, the load transfer ratio the coated cylinder (sphere) assemblage with core of material one and coating of material two attains the lower bound (3.11).

For $\kappa_1 < \kappa_2$ and $1 > \theta_2 > 0$ it follows from Eq. (3.11) that composites amplify the imposed macroscopic stress. Indeed, since $M^2 > 1$ for $\kappa_2 > \kappa_1$ and $1 > \theta_2 > 0$, it is clear that every composite has points inside material two where the magnitude of the local hydrostatic stress is greater than the magnitude of the applied macroscopic stress.

4. Lower bounds on the local hydrostatic stress

In this section, we establish lower bounds on the hydrostatic stress field inside each material. The fourth order identity is denoted by $\mathbf{I}$ and $\mathbf{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The projection onto the hydrostatic part of the stress is denoted by $\mathbf{\Pi}^{\mathrm{H}}$ and is given explicitly by

$$\mathbf{\Pi}^{\mathrm{H}}_{ijkl} = \frac{1}{d}\,\delta_{ij}\delta_{kl} \quad \text{and} \quad \mathbf{\Pi}^{\mathrm{H}}\sigma(\mathbf{x}) = \sigma_{\mathrm{H}}(\mathbf{x})I. \tag{4.1}$$

The projection onto the deviatoric part of the stress is given by $\mathbf{\Pi}^{\mathrm{D}} = \mathbf{I} - \mathbf{\Pi}^{\mathrm{H}}$. The isotropic elasticity tensor associated with each component material is written as

$$C^{\mathrm{i}} = 2\mu^i \mathbf{\Pi}^{\mathrm{D}} + d\kappa^i \mathbf{\Pi}^{\mathrm{H}} \quad \text{for } i = 1, 2, \tag{4.2}$$

where $d = 2$ for planar elastic problems and $d = 3$ for the three-dimensional problem.

For any symmetric $d \times d$ stress field $\eta(\mathbf{x})$ defined on Q, one has

$$\langle \chi_2(\mathbf{x}) \mathbf{\Pi}^{\mathrm{H}}(\sigma(\mathbf{x}) - \eta(\mathbf{x})) : (\sigma(\mathbf{x}) - \eta(\mathbf{x})) \rangle \geqslant 0. \tag{4.3}$$

Setting η equal to a constant stress $\overline{\eta}$ one obtains

$$\langle \chi_2(\mathbf{x})\mathbf{\Pi}^{\mathrm{H}}\sigma(\mathbf{x}) : \sigma(\mathbf{x})\rangle \geqslant 2\mathbf{\Pi}^{\mathrm{H}}\overline{\eta} : \langle \chi_2(\mathbf{x})\sigma(\mathbf{x})\rangle - \theta_2\mathbf{\Pi}^{\mathrm{H}}\overline{\eta} : \overline{\eta}. \tag{4.4}$$

Optimizing over $\overline{\eta}$ gives

$$\langle \chi_2(\mathbf{x})\mathbf{\Pi}^{\mathrm{H}}\sigma(\mathbf{x}) : \sigma(\mathbf{x})\rangle \geqslant \frac{1}{\theta_2}\,\mathbf{\Pi}^{\mathrm{H}}\langle \chi_2(\mathbf{x})\sigma(\mathbf{x})\rangle : \langle \chi_2(\mathbf{x})\sigma(\mathbf{x})\rangle. \tag{4.5}$$

It now easily follows from Eq. (4.1) that

$$\langle \chi_2(\mathbf{x})|\sigma_{\mathrm{H}}(\mathbf{x})|^2\rangle \geqslant \frac{1}{\theta_2}\,|\langle \chi_2(\mathbf{x})\sigma_{\mathrm{H}}(\mathbf{x})\rangle|^2. \tag{4.6}$$

Division of both sides of Eq. (4.6) by σ_0^2 gives

$$\langle \chi_2(\mathbf{x})|R_{\mathrm{H}}(\mathbf{x})|^2\rangle \geqslant \frac{1}{\theta_2}\,|\langle \chi_2(\mathbf{x})R_{\mathrm{H}}(\mathbf{x})\rangle|^2. \tag{4.7}$$

From Eqs. (2.5) and (4.7) one obtains

$$\langle \chi_2(\mathbf{x})|R_{\mathrm{H}}(\mathbf{x})|^2\rangle \geqslant \frac{(\kappa^2)^2(1-\kappa^1(C^{\mathrm{e}})^{-1}I : I)^2}{\theta_2(\kappa^2-\kappa^1)^2}. \tag{4.8}$$

For p and q such that $p \geqslant 1$ and $1/p + 1/q = 1$, one applies Hölder's inequality to obtain

$$\theta_2^{1/q}\langle \chi_2(\mathbf{x})|R_{\mathrm{H}}(x)|^{2p}\rangle^{1/p} \geqslant \langle \chi_2(\mathbf{x})|R_{\mathrm{H}}(\mathbf{x})|^2\rangle, \tag{4.9}$$

and it follows that

$$\langle \chi_2(\mathbf{x})|R_{\mathrm{H}}(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \frac{\theta_2^{1/p}}{\theta_2^2}\left(\frac{(\kappa^2)(1-\kappa^1(C^{\mathrm{e}})^{-1}I : I)}{(\kappa^2-\kappa^1)}\right)^2 \tag{4.10}$$

for $1 \leqslant p \leqslant \infty$.

Similar arguments give the lower bound

$$\langle \chi_1(\mathbf{x})|R_{\mathrm{H}}(\mathbf{x})|^2\rangle \geqslant \frac{(\kappa^1)^2(1-\kappa^2(C^{\mathrm{e}})^{-1}I : I)^2}{\theta_1(\kappa^1-\kappa^2)^2}, \tag{4.11}$$

and it follows that

$$\langle \chi_1(\mathbf{x})|R_{\mathrm{H}}(x)|^{2p}\rangle^{1/p} \geqslant \frac{\theta_1^{1/p}}{\theta_1^2}\left(\frac{(\kappa^1)(1-\kappa^2(C^{\mathrm{e}})^{-1}I : I)}{(\kappa^1-\kappa^2)}\right)^2 \tag{4.12}$$

for $1 \leqslant p \leqslant \infty$. It is emphasized that the bounds given by (4.10) and (4.12) hold for all anisotropic composites.

Explicit bounds on the contraction $(C^{\mathrm{e}})^{-1}I : I$ follow immediately from the work of Kantor and Bergman (1984) and are given by

$$(\kappa_{\mathrm{HS}}^{+})^{-1} \leqslant (C^{\mathrm{e}})^{-1}I : I \leqslant (\kappa_{\mathrm{HS}}^{-})^{-1}, \tag{4.13}$$

where κ^-_{HS} and κ^+_{HS} are the Hashin and Shtrikman (1963) bulk modulus bounds given by

$$\kappa^+_{HS} = \kappa^1\theta_1 + \kappa^2\theta_2 - \left(\frac{\theta_1\theta_2(\kappa^2-\kappa^1)^2}{\kappa^1\theta_2 + \kappa^2\theta_1 + 2\,[(d-1)/d]\,\mu^1}\right) \tag{4.14}$$

and

$$\kappa^-_{HS} = \kappa^1\theta_1 + \kappa^2\theta_2 - \left(\frac{\theta_1\theta_2(\kappa^2-\kappa^1)^2}{\kappa^1\theta_2 + \kappa^2\theta_1 + 2\,[(d-1)/d]\,\mu^2}\right). \tag{4.15}$$

The work of Kantor and Bergman (1984) shows that the bounds given by Eqs. (4.13) hold both for the well-ordered case $\kappa^1 > \kappa^2$, $\mu^1 > \mu^2$ and the non-well-ordered case $\kappa^1 < \kappa^2$, $\mu^1 > \mu^2$.

For the well-ordered case, $\kappa^1 > \kappa^2$ one applies inequality (4.13) to Eqs. (4.10) and (4.12) to find that

$$\langle \chi_2(\mathbf{x})|R_H(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \theta_2^{1/p} \times \left(\frac{\kappa^1\kappa^2((\kappa^1)^{-1} - (\kappa^+_{HS})^{-1})}{\theta_2(\kappa^2-\kappa^1)}\right)^2 = \theta_2^{1/p}(L^2)^2 \tag{4.16}$$

and

$$\langle \chi_1(\mathbf{x})|R_H(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \theta_1^{1/p} \times \left(\frac{\kappa^1\kappa^2((\kappa^2)^{-1} - (\kappa^-_{HS})^{-1})}{\theta_1(\kappa^1-\kappa^2)}\right)^2 = \theta_1^{1/p}(L^1)^2 \tag{4.17}$$

and bounds (3.6) and (3.7) follow.

To obtain Eq. (3.8) we recall Eq. (3.10) for $r = \infty$ to see that

$$L^1 \leqslant \| |R_H(\mathbf{x})| \|_{L^\infty(Q_1)} \leqslant \| |R_H(\mathbf{x})| \|_{L^\infty(Q)} \tag{4.18}$$

and Eq. (3.8) follows.

For the non-well-ordered case, $\kappa^1 < \kappa^2$ one applies inequality (4.13) to Eqs. (4.10) and (4.12) to find that

$$\langle \chi_2(\mathbf{x})|R_H(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \theta_2^{1/p} \times \left(\frac{\kappa^1\kappa^2((\kappa^1)^{-1} - (\kappa^-_{HS})^{-1})}{\theta_2(\kappa^2-\kappa^1)}\right)^2 = \theta_2^{1/p}(M^2)^2 \tag{4.19}$$

and

$$\langle \chi_1(\mathbf{x})|R_H(\mathbf{x})|^{2p}\rangle^{1/p} \geqslant \theta_1^{1/p} \times \left(\frac{\kappa^1\kappa^2((\kappa^2)^{-1} - (\kappa^+_{HS})^{-1})}{\theta_1(\kappa^1-\kappa^2)}\right)^2 = \theta_1^{1/p}(M^1)^2 \tag{4.20}$$

and bounds (3.6) and (3.7) follow.

To obtain Eq. (3.11) we recall Eq. (3.6) for $r = \infty$ to see that

$$M^2 \leqslant \| |R_H(\mathbf{x})| \|_{L^\infty(Q_2)} \leqslant \| |R_H(\mathbf{x})| \|_{L^\infty(Q)} \tag{4.21}$$

and Eq. (3.11) follows.

5. Optimality

In this section, it is shown that the lower bounds presented in Section 3 are attained by the hydrostatic component of the stress fields inside the Hashin–Shtrikman (1962) coated sphere and cylinder assemblages. The coated cylinder assemblage is constructed as follows. A space filling configuration of disks of different sizes ranging down to the infinitesimal is placed inside the unit square Q. Each disk is then partitioned into an annulus called the coating and a concentric disk called the core. The area fractions of coating and core are the same for all disks. The unit square Q filled with the coated cylinder assemblage is illustrated in Fig. 2. The construction of the coated sphere assemblage follows the same pattern. A space filling configuration of spheres is placed inside the unit cube. Each sphere is partitioned into a spherical shell called the coating and a concentric sphere called the core. Here, the volume fractions of coating and core are the same for every sphere.

The explicit formula for the bulk modulus for the coated sphere construction was derived in Hashin (1962). The formula for the bulk modulus for the coated cylinders construction was given by Hashin and Rosen (1964). It is well known from the work of Hashin and Shtrikman (1963) that the associated effective bulk moduli for the coated sphere assemblages attain the bulk modulus bounds κ_{HS}^{-} and κ_{HS}^{+}. The analogous statements for the coated cylinder assemblages can be found in the work of Hashin and Rosen (1964). It is pointed out that the hydrostatic stress fields are constant inside the core phase and inside the coating phase for the coated sphere and cylinder assemblages.

For assemblages with core of material one and coating of material two, the effective bulk modulus is given by κ_{HS}^{-}, the hydrostatic stress field inside the core is given by $\sigma_0 L^1$ and the hydrostatic stress field inside the coating is given by $\sigma_0 M^2$. For assemblages with core of material two and coating of material one, the effective bulk modulus is given by κ_{HS}^{+}, the hydrostatic stress field inside the core is given by $\sigma_0 L^2$ and the hydrostatic stress field inside the coating is given by $\sigma_0 M^1$. From these observations it is evident that bounds (3.6), (3.7), (3.9) and (3.10) are attained by the

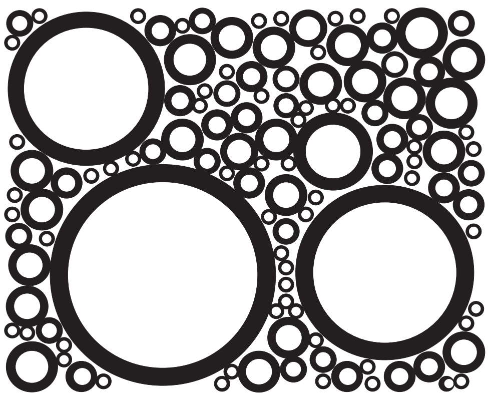

Fig. 2. The unit square is filled with the Hashin–Shtrikman coated cylinder assemblage.

hydrostatic stress fields inside the coated sphere and coated cylinder assemblages. For $\kappa^1 > \kappa^2$, one checks that $L^1 > M^2$ and it follows that coated sphere and cylinder assemblages with a core of material one and coating of material two has a hydrostatic stress field that attains the lower bound (3.8). For $\kappa^1 < \kappa^2$, one checks that $L^1 < M^2$ and it follows that coated sphere and cylinder assemblages with a core of material one and coating of material two has a hydrostatic stress field that attains the lower bound (3.11).

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